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Global defensive alliances in star graphs[☆]

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ABSTRACT

A defensive alliance in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ satisfying the condition that, for each $v \in S$, at least one half of its closed neighbors are in S. A defensive alliance S is called a critical defensive alliance if any vertex is removed from S, then the resulting vertex set is not a defensive alliance any more. An alliance S is called global if every vertex in $V(G) \setminus S$ is adjacent to at least one member of the alliance S. In this paper, we shall propose a way for finding a critical global defensive alliance of star graphs. After counting the number of vertices in the critical global defensive alliance, we can derive an upper bound to the size of the minimum global defensive alliances in star graphs.

1. Introduction

Let $G = (V, E)$ be an undirected graph, where $V(G)$ and $E(G)$ are the vertex and edge, respectively, sets of G. For simplicity, we also use V and E to represent $V(G)$ and $E(G)$, respectively, when only one graph is mentioned. All graphs considered in this paper are finite, undirected, without loops and multiple edges. For any vertex $v \in V$ and a set $S \subseteq V$, the *open neighborhood* of v in S is the set $N_S(v) = \{u \in S | uv \in E\}$. The *closed neighborhood* of v in S is $N_S[v] = N_S(v) \bigcup \{v\}$. If $S = V$, then we simply write $N(v)$ and $N[v]$ rather than $N_V(v)$ and $N_V[v]$, respectively. A vertex $u \in N(v)$ (respectively, $u \in N[v]$) is called an *open neighbor* (respectively, a *closed neighbor*) of v. A nonempty set $S \subseteq V$ of vertices is called a *defensive alliance* (respectively, *strong defensive alliance*) if and only if for every $v \in S$, $|N_S[v]| \geq |N_{V\setminus S}(v)|$ (respectively, $|N_S[v]| > |N_{V\setminus S}(v)|$), i.e., at least one half of v's closed neighbors are in S, where $V \setminus S$ denotes the difference of sets V and S [11,13]. The *defensive alliance number* $a(G)$ (respectively, *strong defensive alliance number* $\hat{a}(G)$) is the minimum cardinally among all defensive alliances (respectively, strong defensive alliances). An alliance S is called *global* if it forms a dominating set (i.e., every vertex in $V \setminus S$ is adjacent to at least one vertex in S). The *global defensive alliance number* of G, denoted by $\gamma_d(G)$, is the cardinality of a global defensive alliance with minimum size. Similarly, $\gamma_{\hat{d}}(G)$ stands for the *strong global defensive alliance number* of G. A global defensive alliance S is called a *critical global defensive alliance* if any vertex is removed from S, then the resulting vertex set is not a global defensive alliance any more. Note that a minimum global defensive alliance is a critical global defensive alliance. However, the converse might not be true.

The original study and motivated definition of defensive alliances in graphs were given in [13]. In that paper, Kristiansen, Hedetniemi and Hedetniemi introduced some mathematical properties of alliances and found that $a(G) \leq \lceil n/2 \rceil$ and $\hat{a}(G) = \lfloor n/2 \rfloor + 1$ for a general graph G where n is the number of vertices in G. Furthermore, bounds of defensive alliance numbers for cycles C_n, paths P_n, tree T, wheels W_n, grids $G_{m,n}$, complete graphs K_n, complete bipartite graphs $K_{r,s}$, and d-regular graphs R_d are also proposed in [13] which are listed in Table 1. In Table 1, the defensive alliance numbers of two

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Table 1
Previous results on defensive alliance numbers.

Graph classes	$a(G)$	$\hat{a}(G)$
C_n, P_n	$a(C_n) = a(W_n) = 2$	$\hat{a}(C_n) = \hat{a}(P_n) = 2, n \geq 3$
T, W_n [13]	$a(P_n) = a(T) = 1$	$\hat{a}(T) \leq \lvert V(T)\rvert, \hat{a}(W_n) = \lceil n/2 \rceil + 1$
$K_n, K_{r,s}$ [13]	$a(K_n) = \lceil n/2 \rceil$	$\hat{a}(K_n) = \lfloor n/2 \rfloor + 1$
	$a(K_{r,s}) = \lfloor r/2 \rfloor + \lfloor s/2 \rfloor, 2 \leq r \leq s$	$\hat{a}(K_{r,s}) = \lceil r/2 \rceil + \lceil s/2 \rceil, 2 \leq r \leq s$
$G_{m,n}$ [13]	$a(G_{m,n}) = 1$ if $\min\{m, n\} = 1$	$\hat{a}(G_{m,n}) = 2$ if $\min\{m, n\} < 3$
	$a(G_{m,n}) = 2$ if $\min\{m, n\} \geq 2$ $\hat{a}(G_{m,n}) = 4$ if $\min\{m, n\} \geq 4$	$\hat{a}(G_{m,n}) = 3$ if $\min\{m, n\} = 3$
R_d [13]	$a(R_i) = i, i = 1$ *or* $2, a(R_3) = 2$ $a(R_d) = girth(R_d), d = 4$ *or* 5	$\hat{a}(R_i) = 2, i = 1$ *or* 2 $\hat{a}(R_d) =$girth$(R_d), d = 3$ *or* 4
$\mathcal{L}(G)$ [19]	$\lceil (\delta_n + \delta_{n-1} - 1)/2 \rceil \leq a(\mathcal{L}(G)) \leq \delta_1$	$\lceil (\delta_n + \delta_{n-1})/2 \rceil \leq \hat{a}(\mathcal{L}(G)) \leq \delta_1$
S_{δ_1,δ_2} [19]	$a(S_{\delta_1,\delta_2}) = \lceil (\delta_1 + \delta_2 - 1)/2 \rceil$	$\hat{a}(S_{\delta_1,\delta_2}) = \lceil (\delta_1 + \delta_2)/2 \rceil$

Table 2
Previous results on global defensive alliance numbers.

Graph classes	$\gamma_d(G)$	$\gamma_{\hat{d}}(G)$
K_n [11]	$\left\lfloor \frac{n+1}{2} \right\rfloor$	$\left\lceil \frac{n+1}{2} \right\rceil$
$K_{r,s}$ [11]	$\gamma_d(K_{1,s}) = \left\lfloor \frac{s}{2} \right\rfloor + 1$ $\gamma_d(K_{r,s}) = \left\lceil \frac{r}{2} \right\rceil + \left\lfloor \frac{s}{2} \right\rfloor$ if $r, s \geq 2$	$\gamma_{\hat{d}}(K_{r,s}) = \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor$
$\mathcal{B}$ [11]	$\gamma_d(\mathcal{B}) \geq \frac{2n}{\Delta+3}$	$\gamma_{\hat{d}}(\mathcal{B}) \geq \frac{2n}{\Delta+2}$
R_4 [11]	$\gamma_d(R_4) \geq \frac{n}{3}$	
T [11]	$\frac{n+2}{4} \leq \gamma_d(T) \leq \frac{3n}{5}$	$\frac{n+2}{3} \leq \gamma_{\hat{d}}(T) \leq \frac{3n}{4}$
G [11,16]	$\gamma_d(G) \geq \frac{\sqrt{4n+1}-1}{2}$	$\gamma_{\hat{d}}(G) \geq \sqrt{n}$
	$\gamma_d(G) \geq \lceil \frac{n}{\lambda+2} \rceil$	$\gamma_{\hat{d}}(G) \geq \lceil \frac{n}{\lambda+1} \rceil$
$\mathcal{P}$ [5]	$\gamma_d(\mathcal{P}) \geq \left\lceil \frac{n+6}{6} \right\rceil$	$\gamma_{\hat{d}}(\mathcal{P}) \geq \left\lceil \frac{n+6}{5} \right\rceil$
$\mathcal{L}(G)$ [19]	$\gamma_d(\mathcal{L}(G)) \geq \lceil \frac{2m}{\delta_1+\delta_2+1} \rceil$	$\gamma_{\hat{d}}(\mathcal{L}(G)) \geq \lceil \frac{2m}{\delta_1+\delta_2} \rceil$

other graphs: semiregular graphs S_{δ_1,δ_2} and line graphs $\mathcal{L}(G)$ are also included which are proposed in [19]. Note that, in Table 1, $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_n$ are the degrees of the n vertices in graph G and the degree of any vertex in semiregular graph S_{δ_1,δ_2} is either δ_1 or δ_2. Note that δ_n is the maximum degree of a graph and δ_{n-1} might equal to δ_n.

Comparing with the defensive alliance problem, the global defensive alliance problem is much harder. The problem of finding a minimum global defensive alliance is NP-complete on general graphs [3,7]. Several bounds on different graph types of global (strong) defensive alliance numbers were obtained in [5,11,16,19] which are summarized in Table 2. In Table 2, n and m are the numbers of vertices and edges, respectively, in graph G, Δ is the maximum degree of graph G, λ is the spectral radius of graph G, and bipartite graphs, planar graphs, and general graphs are denoted by $\mathcal{B}$, $\mathcal{P}$, and G, respectively. For other different types of alliances, the reader is referred to [5,10,19] for (strong) defensive alliances, [6,17,18] for (strong) offensive alliances, and [2] for powerful alliances. A generalized version of global defensive alliance problem called *defensive (offensive) r-alliance problem* was introduced in [7] in which every vertex v in an offensive set S is asked to have $|N_S[v]|/2 + r - 1$ closed neighbors in S. Besides, defensive alliances have related concept in the context of coalition, monopolies, and distributed computing [9,14,15]. A relatively new concept of alliances (defensive or offensive) in network graphs has recently attracted a great deal of attention due to some interesting applications in web communities [8,12] and fault-tolerant computing [15,20,21]. For fault-tolerant computing in network graphs, processors are partitioned into two alliances to adopt majority voting when there is a conflict on distributed data [15].

In this paper, we restrict the global defensive alliance problem to a family of symmetric graphs: the well-known star graphs. Star graphs were proposed as an attractive alternative to hypercubes with many nice topological properties [4]. Star graphs have many superior advantages over hypercubes such as a smaller degree and diameter. Particularly, considering the regularity and the underlying algebraic structure, star graphs are good candidates for interconnecting vertices of a network. In [1], Arumugam and Kala derived an upper bound of the domination problem on star graphs. Inspired by this concept, in this paper, we shall propose upper bounds for the global defensive alliance numbers and the global strong defensive alliance numbers on star graphs. Star graphs are regular graphs and have various degrees. This means that our results also provide upper bounds for various regular graphs simultaneously if there are $n!$ vertices in an $(n-1)$-regular graph.

The remaining part of this paper is organized as follows. Section 2 introduces the basic terminology and notation. In Section 3, we propose a way for finding a critical global defensive alliance of an even star graph. Then, by counting the number of vertices in the finding critical global defensive alliance, we derive an upper bound to the size of the minimum global defensive alliance on an even star graph. In Section 4, a way for finding a critical global defensive alliance of an odd

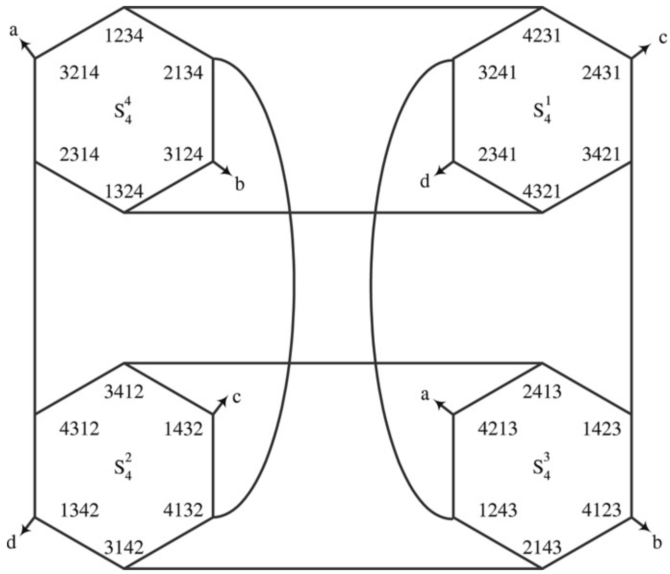

Fig. 1. The four-dimensional star graph S_4.

star graph is proposed. Then, we derive an upper bound to the size of the minimum global defensive alliance on a star graph with odd dimension. Finally, concluding remarks are given in the last section.

2. Preliminaries

A set $S \subseteq V$ is a *dominating set* of graph $G = (V, E)$ if for every vertex $u \in V \setminus S$ there exists a vertex $v \in S$ such that u is adjacent to v. We also say that v *dominates* u and u is *dominated* by v. A dominating set S is said to be a *total dominating set* if every vertex $u \in S$ is adjacent to another vertex in S. The *total domination number*, denoted by $\gamma_t(G)$, of graph G is the cardinality of a total dominating set which has the minimum number of vertices among all total dominating sets. An *induced subgraph* $< S >$ is the maximal subgraph of G with vertex set S. Let $\deg_G(v)$ denote the degree of vertex v in G and $\delta(G)$ denote the minimum degree of G, i.e., $\delta(G) = \min_{v \in V}\{\deg_G(v)\}$. A *permutation* is a sequence of elements in which no element appears more than once. Let $N = \{1, 2, \dots, n\}$ and $[p_1, p_2, \dots, p_n]$ be a permutation, where $p_i \in N$ for all $1 \le i \le n$. The *n-dimensional star graph* (n-star for short), denoted by S_n, is an undirected graph consisting of $n!$ vertices labeled by distinct permutations $[p_1, p_2, \dots, p_n]$. An n-star is called an *odd dimensional* (respectively, *even dimensional*) star graph if n is odd (respectively, even). Let $P_v = [p_1, p_2, \dots, p_n]$ denote the label of vertex v and $P_v(i)$ denote the ith number of P_v, namely p_i. We also use $P_v^{-1}(j)$ to denote the position of symbol j in P_v. For example, if $P_v = [3142]$, then $P_v(3) = 4$ and $P_v^{-1}(4) = 3$. Note that, for brevity, we always omit the comma between two symbols in a label if it will not make confusion. That is, for example, $P_v = [3142]$ rather than $P_v = [3, 1, 4, 2]$. Two vertices are connected by an edge in a star graph if and only if the label of one vertex can be obtained by swapping the first symbol (conventionally, the leftmost) and the ith symbol of the other vertex, where $2 \le i \le n$. Fig. 1 depicts S_4 which contains 24 vertices and vertices [1234] and [4231] are neighbors due to their labels differ only at the first position and the last position. Note that an n-star is an edge-symmetric and vertex-symmetric regular graph of degree $n - 1$. Thus, each vertex in S_n has $n - 1$ neighbors.

The class of star graphs has a highly recursive structure. A *k-dimensional substar*, or *k-substar*, is an induced subgraph of S_n in which the vertex set of a k-substar contains all vertices having the same values in $n - k$ specific positions of their labels. For example, the induced subgraph of the set of vertices $\{v | P_v(n) = 1\}$ forms an $(n - 1)$-substar. Note that each of sets $\{v | P_v(i) = j\}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$, is also an $(n - 1)$-substar. Specifically, we use $S_{n,i}$ to denote the $(n - 1)$-substar which contains the set of vertices $\{v | P_v(n) = i\}$ for $i = 1, 2, \dots, n$.

The following two vertex sets A_i and B_{ij} which are based on S_n will be used to find the bounds of the global defense alliance problem on star graphs.

$$A_i = \begin{cases} \{v \in V(S_n) | P_v(1) = i + 1 \text{ and } P_v(n) = i\} & \text{if } i \text{ is odd and} \\ \{v \in V(S_n) | P_v(1) = i - 1 \text{ and } P_v(n) = i\} & \text{if } i \text{ is even,} \end{cases}$$

$$B_{ij} = \begin{cases} \{v \in V(S_n) | P_v(k) = i \text{ and } P_v(n) = i + 1\} & \text{if } i \text{ is odd and} \\ \{v \in V(S_n) | P_v(k) = i \text{ and } P_v(n) = i - 1\} & \text{if } i \text{ is even} \end{cases}$$

where $k = \lfloor \frac{n}{2} \rfloor + j + 1, j = 1, 2, \dots, \lceil \frac{n}{2} \rceil - 2$ and $i = 1, 2, \dots, n$ if n is even and $i = 1, 2, \dots, n - 1$ if n is odd. In the remaining part of our discussion, we shall use the collecting sets of A_i and B_{ij} frequently. Thus, we use A_o and B_o to stand for the sets $\bigcup_{i=1}^{n-1} A_i$ and $\bigcup_{\substack{1 \le i \le n-1 \\ 1 \le j \le (n-3)/2}} B_{ij}$, respectively, for odd dimensional star graphs S_n with $n \ge 5$. For even dimensional star

Table 3
A_i of S_8.

A_1	{2xxxxxx1}
A_2	{1xxxxxx2}
A_3	{4xxxxxx3}
A_4	{3xxxxxx4}
A_5	{6xxxxxx5}
A_6	{5xxxxxx6}
A_7	{8xxxxxx7}
A_8	{7xxxxxx8}

Table 4
B_{ij} of S_8.

B_{11}	{xxxxx1x2}	B_{12}	{xxxxxx12}
B_{21}	{xxxxx2x1}	B_{22}	{xxxxxx21}
B_{31}	{xxxxx3x4}	B_{32}	{xxxxxx34}
B_{41}	{xxxxx4x3}	B_{42}	{xxxxxx43}
B_{51}	{xxxxx5x6}	B_{52}	{xxxxxx56}
B_{61}	{xxxxx6x5}	B_{62}	{xxxxxx65}
B_{71}	{xxxxx7x8}	B_{72}	{xxxxxx78}
B_{81}	{xxxxx8x7}	B_{82}	{xxxxxx87}

graphs S_n, let A_e and B_e denote the sets $\bigcup_{i=1}^{n} A_i$ and $\bigcup_{\substack{1\le i\le n \\ 1\le j\le \frac{n}{2}-2}} B_{ij}$, respectively, for $n \geq 6$. Moreover, let $H_e = A_e \bigcup B_e$ and $H_p = A_o \bigcup B_o$.

For example, let us consider star graphs S_7 and S_8. Star graph S_7 has $A_1, A_2, \dots, A_6$ while star graph S_8 has $A_1, A_2, \dots, A_8$. In both of S_7 and S_8, A_1 contains all vertices with $P_v(1) = 2$ and $P_v(n) = 1$, A_2 contains all vertices with $P_v(1) = 1$ and $P_v(n) = 2$, etc. For convenience, we use $\{\mathrm{xx}\cdots j\mathrm{xx}\cdots\}$ to denote the set of vertices with $P_v(i) = j$. Thus, in S_7, {2xxxxx1} denotes the set of vertices in A_1. S_7 has $B_{11}, B_{12}, B_{21}, B_{22}, \dots, B_{62}$ while S_8 has $B_{11}, B_{12}, B_{21}, B_{22}, \dots, B_{82}$. Sets B_{11} and B_{12} of S_7 contain the sets of vertices {xxxx1x2} and {xxxxx12}, respectively. Tables 3 and 4 list all vertices in A_i and B_{ij}, respectively, for S_8.

In accordance with the definition of B_{ij}, we classify swapping operations on vertex v into the following three classes. A swapping operation is called a *left exchange* (respectively, *right exchange*) if the swapping operation occurs at positions 1 and j where $2 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1$ (respectively, $\lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n-1$). If a swapping operation occurs at positions 1 and n, then this swapping operation is called an *end exchange*. A vertex u is called a *left exchange neighbor* (respectively, *right exchange neighbor*), abbreviated *ℓ-neighbor* (respectively, *r-neighbor*), of vertex v if u can be obtained from v by using a left exchange (respectively, right exchange). If u can be obtained from v by using an end exchange, then u is called an *end exchange neighbor* (*e-neighbor* for short) of v. The sets of all ℓ-neighbors, all r-neighbors, and the e-neighbor of v are denoted by $N_\ell(v)$, $N_r(v)$, and $N_e(v)$, respectively. For example, $N_\ell(v) = \{[32456781], [43256781], [53426781], [63452781]\}$, $N_r(v) = \{[73456281], [83456721]\}$, and $N_e(v) = \{[13456782]\}$ if $v = [23456781]$ in S_8.

In [11], Haynes et al. showed that $\gamma_d(G) = \gamma_t(G)$ when the minimum degree of graph G is at least two and the maximum degree is at most three. We can obtain that $\gamma_d(S_3) = \gamma_t(S_3)$ and $\gamma_d(S_4) = \gamma_t(S_4)$. Thus, henceforth, we will always assume that S_n is with $n \geq 5$.

3. Even dimensional star graphs

In this section, we discuss the global defensive alliance problem on even dimensional star graphs. Arumugam and Kala [1] found that a set of vertices labeled with a given symbol at the first position plays a leading role for studying the domination problem on star graphs and proposed the following lemma.

Lemma 1 (*[1]*). *The set* $\{i\mathrm{xx}\cdots\mathrm{xx}\}$ *for some* $i \in \{1, 2, \dots, n\}$ *is a minimum dominating set of* S_n.

Lemma 2. *The set* A_e *is a dominating set of* S_n *for even* $n \geq 6$.

Proof. Since $S_{n,i}$ contains the set of vertices $\{\mathrm{x}\cdots\mathrm{x}i\}$ for $i = 1, 2, \dots, n$, A_i contains all of the vertices with labels beginning with $i+1$ (respectively, $i-1$) if i is an odd (respectively, even) number. By Lemma 1, A_i is a dominating set of $S_{n,i}$. Therefore, $A_e = \bigcup_{i=1}^{n} A_i$ is a dominating set of S_n for even $n \geq 6$. $\square$

Recall that $A_e = \bigcup_{i=1}^{n} A_i$ and $B_e = \bigcup_{\substack{1\le i\le n \\ 1\le j\le \frac{n}{2}-2}} B_{ij}$. To show that $H_e = A_e \bigcup B_e$ is a global defensive alliance of S_n, we need to calculate the number of closed neighbors of each $v \in H_e$ as follows.

Lemma 3. $A_e \bigcap B_e = \emptyset$ *for even* $n \geq 6$.

Proof. Let v be a vertex in B_e. Assume that v is in B_{ij}, where $1 \le i \le n$ and $1 \le j \le \frac{n}{2} - 2$. All we have to prove is that v is not in A_e. Then, depending on i, there are two cases to consider.

Case 1: i is odd.

In this case, by the definition of B_{ij}, $P_v(n) = i + 1$ and $i + 1$ is an even number. Then, by the definition of A_e, a vertex u in A_e with even number $P_u(n) = i + 1$ must have $P_u(1) = i$. However, by the definition of B_{ij} again, $P_v(k) = i$ for some $k \ne 1$. Thus, $P_v(1) \ne i$ and v is not in A_e.

Case 2: i is even.

In this case, with a similar reasoning as Case 1, we can find that $P_v(n) = i - 1$ which is an odd number. By the definition of A_e, a vertex u in A_e with odd number $P_u(n) = i - 1$ must have $P_u(1) = i$. However, by the definition of B_{ij} again, $P_v(k) = i$ for some $k \ne 1$. Thus, $P_v(1) \ne i$ and v is not in A_e. This completes the proof. $\square$

Lemma 4. $\delta(< H_e >) \ge \frac{n}{2} - 1$ *for even* $n \ge 6$.

Proof. Let v be a vertex in H_e. This means that v is in either A_i or B_{ij}. Thus, we have two cases to consider.

Case 1: $v \in A_i$.

If i is odd, then, by definition, $P_v(1) = i + 1$ and $P_v(n) = i$. Vertex v has exactly one e-neighbor, say u, in A_{i+1}. Vertex v has $\frac{n}{2} - 2$ distinct r-neighbors and each of them, say w, is in $B_{i+1,j}$ where $j = P_w^{-1}(i+1) - \frac{n}{2} - 1$. Note that, by Lemma 3, A_e and B_e are disjoint. Thus, $\deg_{<H_e>}(v) = \frac{n}{2} - 1$. Similarly, we can prove that this case holds for even number i.

Case 2: $v \in B_{ij}$.

Since only positions $\frac{n}{2} + j + 1$ and n are fixed in every label of vertices in B_{ij} for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, \frac{n}{2} - 2$, B_{ij} is an $(n-2)$-substar of S_n. Therefore, every vertex in $< B_{ij} >$ has degree $n - 3$. This means that $\deg_{<H_e>}(v) > \frac{n}{2} - 1$ when $n \ge 6$. $\square$

By Lemmas 2 and 4, the set of vertices in H_e forms a global defensive alliance of S_n for $n \ge 6$. We shall further show that H_e is critical.

Theorem 5. *The set of vertices in H_e forms a critical global defensive alliance of S_n for even $n \ge 6$.*

Proof. To prove that H_e is a critical global defensive alliance of S_n, we consider whether there exists a vertex $v \in H_e$ such that $H_e - v$, i.e., removing vertex v from H_e, is still a global defensive alliance of S_n. There are two cases to consider.

Case 1: $v \in A_e$

In this case, we shall prove that $H_e - v$ is not a dominating set of S_n any more. Assume that v is in A_i. Let u be an ℓ-neighbor of v with $P_u(1) = k$ and $P_u(j) \ne k + 1$ (respectively, $k - 1$) if k is odd (respectively, even) for $\frac{n}{2} + 2 \le j \le n - 1$. Clearly, k is not equal to i or $i - 1$ (respectively, $i + 1$) if i is even (respectively, odd).

We first prove that u is not a neighbor of any vertex in $A_e - v$. It is clear that u cannot be a neighbor of any vertex in $A_i - v$. Now we assume on the contrary that u has a neighbor in $A_e - A_i$. Since u is an ℓ-neighbor of v, $P_u(n) = i$. Any ℓ-neighbor or r-neighbor of u must be not in $A_e - A_i$. The e-neighbor of u, say w, is with $P_w(1) = i$ and $P_w(n) = k$. Moreover, $P_w^{-1}(i-1) = P_u^{-1}(i-1) = j$ if i is even and $P_w^{-1}(i+1) = P_u^{-1}(i+1) = j$ otherwise, where j is not equal to 1 or n. By definition, if w is in A_k, then $P_w(1)$ must be equal to $k + 1$ (respectively, $k - 1$) if k is odd (respectively, even). This means that i is equal to $k + 1$ and $k = i - 1$ if k is odd and $k = i + 1$ if k is even. It contradicts that $P_w^{-1}(i-1)$ is not equal to n for the former case and $P_w^{-1}(i+1)$ is not equal to n for the latter case. Therefore, u is not a neighbor of any vertex in $A_e - v$.

Now we prove that u can also not be a neighbor of any vertex in B_e. The reason is that if u is a neighbor of some vertex, say w, in B_e, then $P_u(n) \ne P_w(n)$. Therefore, the only way to obtain w from u is to do an end exchange on u. However, if $P_w(n) = k$ and w is a neighbor of u, then there must exist some $P_w(j) = k + 1$ (respectively, $k - 1$) if k is odd (respectively, even) for $\frac{n}{2} + 2 \le j \le n - 1$ in $B_{k+1, j-\frac{n}{2}-1}$(respectively, $B_{k-1, j-\frac{n}{2}-1}$). This is also impossible for the designated u. Therefore, $N[u] \bigcap H_e = \{v\}$ and $H_e - v$ is not a dominating set of S_n any more.

Case 2: $v \in B_e$

In this case, we shall prove that $H_e - v$ is not a defensive alliance of S_n any more. By definition, v must be an r-neighbor of some vertex, say u, in A_e. By Lemma 4, every vertex in A_e has degree $\frac{n}{2} - 1$. Thus, $\deg_{<H_e - v>}(u) < \frac{n}{2} - 1$ and $H_e - v$ is not a defensive alliance of S_n any more. This completes the proof. $\square$

Theorem 6. $\gamma_d(S_n) \le \frac{n-2}{2n-2} n!$ *for even* $n \ge 6$.

Proof. By Theorem 5, H_e is a critical global defensive alliance of S_n. Clearly, $|H_e|$ is an upper bound of $\gamma_d(S_n)$.

Since each of A_i and B_{ij}, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, \frac{n}{2} - 2$, is an $(n-2)$-substar of S_n, $|A_i| = |B_{ij}| = (n-2)!$. Moreover, all of them are disjoint. Therefore,

$$
\begin{aligned}
|H_e| &= |A_e| + |B_e| \\
&= n \cdot (n-2)! + n \cdot (\frac{n}{2} - 2) \cdot (n-2)! \\
&= \frac{n-2}{2n-2} n! \quad \square
\end{aligned}
$$

Corollary 7. $\gamma_{\hat{d}}(S_n) \le \frac{n^2(n-2)!}{2}$ *for even* $n \ge 6$.

Table 5
A_i^n of S_7.

A_1^7	{2xxxx17}
A_2^7	{1xxxx27}
A_3^7	{4xxxx37}
A_4^7	{3xxxx47}
A_5^7	{6xxxx57}
A_6^7	{5xxxx67}

Table 6
B_{ij}^n of S_7.

B_{11}^7	{xxxx127}
B_{21}^7	{xxxx217}
B_{31}^7	{xxxx347}
B_{41}^7	{xxxx437}
B_{51}^7	{xxxx567}
B_{61}^7	{xxxx657}

Proof. If parameters k, j, and i of B_{ij} are set to $k = \frac{n}{2} + j, j = 1, 2, \ldots, \frac{n}{2} - 1$, and $i = 1, 2, \ldots, n$ for even n, then the degree of any vertex in H_e will be greater than or equal to $\frac{n}{2}$. By a similar argument as Lemma 3, we can prove that A_e and B_e are still disjoint under this rearrangement. Therefore,

$$\begin{aligned} |H_e| &= |A_e| + |B_e| \\ &= n \cdot (n-2)! + n \cdot \left(\frac{n}{2} - 1\right) \cdot (n-2)! \\ &= \frac{n^2(n-2)!}{2}. \quad \Box \end{aligned}$$

4. Odd dimensional star graphs

In this section, we discuss the global defensive alliance problem on odd dimensional star graphs. In the definitions of A_i and B_{ij} for odd dimensional star graph S_n, the range for i is between 1 and $n - 1$. Viewing $S_{n,n}$ as an even dimensional star graph S_{n-1} (i.e., neglecting the last symbol, which is n, in the label of the vertices in $S_{n,n}$), $S_{n,n}$ itself has its own A_i and B_{ij}. We use A_i^n and B_{ij}^n to denote A_i and B_{ij}, respectively, in $S_{n,n}$ to avoid confusing with A_i and B_{ij} in S_n. Tables 5 and 6 list all vertices in A_i^n and B_{ij}^n, respectively, for S_7. The sets A_o^n and B_o^n stand for $\bigcup_{i=1}^{n-1} A_i^n$ and $\bigcup_{\substack{1 \le i \le n-1 \\ 1 \le j \le (n-5)/2}} B_{ij}^n$, respectively. For convenience, let $H_o = H_p \bigcup A_o^n \bigcup B_o^n$.

Lemma 8. *$A_o \bigcup A_o^n$ is a dominating set of S_n for odd $n \ge 5$.*

Proof. By definition, A_o dominates the set of all vertices $\{x \cdots xi\}$, i.e., $S_{n,i}$, for $i = 1, 2, \ldots, n - 1$. By Lemma 2, A_o^n is a dominating set of $S_{n,n}$. This completes the proof. $\Box$

Lemma 9. *$\delta(< H_o >) \ge (n - 1)/2$ for odd $n \ge 5$.*

Proof. Let v be a vertex in H_o. There are four cases to consider.

Case 1: $v \in A_o$.

Clearly, $|N_\ell(v) \bigcap H_o| = 0$, $|N_r(v) \bigcap H_o| = (n - 3)/2$, and $|N_e(v) \bigcap H_o| = 1$. Thus, $\deg_{<H_o>}(v) = 0 + (n - 3)/2 + 1 = (n - 1)/2$.

Case 2: $v \in B_o$.

Since B_{ij} is an $(n - 2)$-substar of S_n. Therefore, by definition, every vertex in $< B_{ij} >$ has degree $(n - 3)$. Assume that $P_v(n) = i + 1$ if i is an odd number and $P_v(n) = i - 1$ otherwise. Thus, by swapping positions 1 and $P_v^{-1}(i)$ of v's label, v has a neighbor in A_{i+1} or A_{i-1} depending on i is odd or even. This implies $\deg_{<H_o>}(v) \ge n - 2 > (n - 1)/2$, for odd $n \ge 5$.

Case 3: $v \in A_o^n$.

By the proof of Case 1 in Lemma 4, $\deg_{<A_o^n \bigcup B_o^n>}(v) = (n-3)/2$. Since the label of v is with $P_v(1) = i+1$, $P_v(n-1) = i$, and $P_v(n) = n$ if i is odd and $P_v(1) = i - 1$, $P_v(n - 1) = i$, and $P_v(n) = n$ otherwise. Therefore, $|N_e(v) \bigcap B_o| = 1$ and $\deg_{<H_o>}(v) = (n - 3)/2 + 1 = (n - 1)/2$.

Case 4: $v \in B_o^n$.

With a similar reasoning as Case 2, we can prove that $\deg_{<H_o>}(v) \ge (n - 1)/2$ for this case. $\Box$

Theorem 10. *The set of vertices in H_o forms a critical global defensive alliance of S_n for odd $n \ge 5$.*

Proof. The proof of this theorem is similar to the proof of Theorem 5. However, there are four cases to consider: (1) $v \in A_o$, (2) $v \in A_o^n$, (3) $v \in B_o$, and (4)$v \in B_o^n$. For Cases 1 and 2, we shall prove that $H_o - v$ is not a dominating set any more. For Cases 3 and 4, the set $H_o - v$ is not a defensive alliance of S_n.

Case 1: $v \in A_o$

Let u be an ℓ-neighbor of v with $P_u(1) = k, k \neq n$, and $P_u(j) \neq k + 1$ (respectively, $k - 1$) if k is odd (respectively, even) for $(n + 3)/2 \leq j \leq n - 1$. With a similar argument as Case 1 in Theorem 5, we can prove that $N[u] \bigcap H_p = \{v\}$. Since $P_u(1) \neq n, N[u] \bigcap (A_o^n \bigcup B_o^n) = \emptyset$. Therefore, $H_o - v$ is not a dominating set of S_n.

Case 2: $v \in A_o^n$

Since $S_{n,n}$ is an even $(n-1)$-substar, by Theorem 5, there exists a vertex $u \in N_{S_{n,n}}(v)$ such that $N_{S_{n,n}}[u] \bigcap (A_o^n \bigcup B_o^n) = \{v\}$. Furthermore, we select the vertex u with $P_u(1) = k$ and $P_u(j) \neq k + 1$ (respectively, $k - 1$) if k is odd (respectively, even) for $(n + 3)/2 \leq j \leq n - 1$. Recall that $P_u(n) = n$. Under the above constraint set on u, $N_{S_n}[u] \bigcap H_p = \emptyset$. Thus, in this case, $H_o - v$ is also not a dominating set of S_n.

Case 3: $v \in B_o$

By definition, v must be an r-neighbor of some vertex, say u, in A_o. By the proof of Case 3 in Lemma 9, every vertex in A_o has degree $(n - 1)/2$ in H_o. Hence, $\deg_{<H_o - v>}(u) < (n - 1)/2$ and $H_o - v$ is not a defensive alliance of S_n any more.

Case 4: $v \in B_o^n$

We consider this case only when odd $n \geq 7$ since $B_o^n = \emptyset$ when $n = 5$. Let v be an r-neighbor of some vertex, say u, in A_o^n with $P_u(n - 1) = P_v(n - 1)$. Then by a similar reasoning as Case 3, we can obtain that $\deg_{<H_o - v>}(u) < (n - 1)/2$ and $H_o - v$ is not a defensive alliance of S_n any more. This completes the proof. □

Theorem 11. $\gamma_d(S_n) \leq \frac{n^2-2n-1}{2n-4} \cdot (n-1)!$ *for odd* $n \geq 5$.

Proof. By Theorem 10, H_o is a critical global defensive alliance of S_n. It is obvious that $|H_o|$ is an upper bound of $\gamma_d(S_n)$ for odd $n \geq 5$.

Since sets A_o, B_o, A_o^n, and B_o^n are pairwise disjoint, $|H_o|$ can be derived as follows.

$$\begin{aligned}
|H_o| &= |A_o| + |B_o| + \left| A_o^n \bigcup B_o^n \right| \\
&= (n-1) \cdot (n-2)! + (n-1) \cdot \frac{n-3}{2} \cdot (n-2)! + \frac{n-3}{2n-4} \cdot (n-1)! \\
&= \frac{n-1}{2}(n-1)! + \frac{n-3}{2n-4} \cdot (n-1)! \\
&= \frac{n^2-2n-1}{2n-4} \cdot (n-1)!. \quad \square
\end{aligned}$$

Corollary 12. $\gamma_{\hat{d}}(S_n) \leq \frac{n^2-2n-1}{2n-4} \cdot (n-1)!$ *for odd* $n \geq 5$.

5. Concluding remarks

In this paper, we derive upper bounds $\gamma_d(S_n)$ and $\gamma_{\hat{d}}(S_n)$ for n-dimensional star graphs. Clearly, the bound we proposed for even dimensional star graphs is tight since our proposed H_e for star graph S_4 is a minimum total dominating set. In [11], Haynes et al. derived a lower bound $\gamma_d(G) \geq n/3$ for 4-regular graphs of n vertices. Thus, for star graph S_5, $\gamma_d(S_5) \geq 120/3 = 40$. It is clear that this bound is not tight for S_5. However, our derived upper bound for S_5 is $\gamma_d(S_5) \leq 56$. Thus, to determine whether our proposed H_e and H_o are minimum or not will be our future work.

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